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SHORT CUTS IN CREEP BUCKLING ANALYSIS†

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Abstract—It is shown that the creep buckling of slender or thin-walled structures whose material deforms elastically as well as in consequence of time-hardening creep can be analyzed first as if the deformations were due entirely to steady creep. The results of this analysis can be easily modified to account for time-hardening creep and this modification does not involve any approximation when the time-hardening creep strain rate is expressed as the product of a function of stress by a function of time. The effect of simultaneous linearly elastic deformations can be taken into account by multiplying the critical time obtained for steady creep by a numerical factor. This correction involves no farther-reaching approximations than the usual steady-creep buckling analysis itself.

NOTATION

a	amplitude of initial deviations
A	total cross-sectional area of column
b	amplitude of additional elastic displacements
c	amplitude of steady creep displacements
C	constant in creep law (4)
E	Young's modulus of elasticity
f	function of x and y
F	operator
g	operator
G	shear modulus
h	operator; also distance between flanges of idealized column
H	operator
I	moment of inertia of cross section of column
J_2	second invariant of stress deviation tensor defined in (8)
k	factor of proportionality in creep law (49)
K	constant defined in (42)
L	length of column
m	exponent in creep law (22)
n	exponent in creep law (49)
p	scalar
P	axial compressive load acting on column
P_E	Euler's buckling load for column [see (47)]
q	exponent of time in time-hardening creep law
r	scalar

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s_{ij}	ijth component of stress deviation tensor
t	time measured from instant of load application
t_{cr}	critical time
t_E	Euler time defined in (61)
w	additional displacement normal to surface of structure; also non-dimensional additional lateral displacement of column [see (32)]
w_0	initial deviation normal to surface from perfect shape of structure
w_{tot}	total non-dimensional lateral displacement (elastic plus creep) of column
x	coordinate on surface of thin structure; also non-dimensional axial coordinate for column [see (35)]
x^*	axial coordinate for column
y	coordinate on surface of thin structure; also additional lateral displacement of column
y_0	initial deviation of center line of imperfect column from straight axis
γ	amplitude of total displacements of column [see (51)]
δ_{ij}	Kronecker's delta defined in (7)
$\Delta\kappa$	change in curvature of column
ε_E	Euler strain defined in (61)
ε_{ij}	ijth component of strain tensor
$\dot{\varepsilon}_{nom}$	nominal strain rate defined in (61)
ν	Poisson's ratio
ρ	intensity of loading
σ_E	Euler stress defined in (61)
σ_{ij}	ijth component of stress tensor
Φ, ψ, Ω	operators

Subscripts:

cr	critical
e	external (convex) side of column
i	internal (concave) side of column
0	refers to conditions at instant of load application
1	refers to conditions at time t_1

STATEMENT OF THE PROBLEM

MANY theoretical analyses of creep buckling have been published in the last few years on the basis of the assumption that the deformations of the thin or slender structural elements are due exclusively to secondary, or steady, creep. In reality all these elements are capable of deforming elastically at the same time when they creep and often the creep deformations of the material are due to primary creep rather than to secondary creep. Of course, inclusion of elasticity and secondary creep in the study complicates the calculations.

It is the purpose of the present report to show that in many cases of practical importance it is permissible to carry out the major portion of the analytical work on the basis of the assumption that the deformations are due exclusively to secondary creep, and to apply corrections for elasticity and primary creep only as the last phase of the treatment of the problem.

The correction here proposed for primary (rather than secondary or steady) creep is rigorous; it does not involve any approximation. The correction for elastic deformations following Hooke's law is just as rigorous as the steady creep buckling analysis itself if the latter is based on a single-term representation of the shape of displacements and if this shape is the eigenfunction of the elastic problem corresponding to the lowest eigenvalue. In the unusual case when the creep buckling analysis is carried out with greater rigor, the multiplying factor here proposed still represents a good approximation in most cases.

This multiplying factor first appeared in the creep buckling literature in papers by Kempner [1] and Hult [2]. It was found as the solution of the differential equation of the column that creeps under the assumption that the deflected shape is the first eigenfunction of the equations of the linearly elastic column in spite of the nonlinear character of the creep law. The factor appears in the same manner in the senior author's report [3] on which the example at the end of the present paper is based. The senior author suggested the same correction factor for rectangular plates under uniaxial compression [4], but without any proof of its correctness. But the factor itself has been used for many years for the calculation of the perfectly elastic structures of civil engineering; in Timoshenko's *Strength of Materials* [5] its discovery is attributed to J. Perry in 1886.

In the present paper the applicability of the correction factor is extended to thin-walled structures of all kinds. The conditions are given under which use of the factor does not introduce any inaccuracies in the analysis beyond those already inherent in the analysis of the creep buckling of the structure in the presence of steady creep.

GOVERNING EQUATIONS

In the case of columns, plates and shells the problem, as a rule, can be stated mathematically as a set of equilibrium equations

$$\Phi(x, y, w_0, w, \rho) = 0 \tag{1}$$

a set of strain–displacement relations

$$\psi(w_0, w, \varepsilon_{ij}) = 0 \tag{2}$$

a set of boundary conditions

$$\Omega(x, y, w) = 0 \quad \text{on boundaries } y = g(x) \tag{3}$$

and the constitutive equation encompassing elastic and time-hardening creep deformations

$$\dot{\varepsilon}_{ij} = (1/2G)\left[\dot{s}_{ij} + \frac{1-2\nu}{3(1+\nu)}\dot{\sigma}_{kk}\delta_{ij}\right] + CJ_2^m s_{ij} t^q. \tag{4}$$

In these equations x and y are suitable coordinates on the surface of the structural element, $w_0 = w_0(x, y)$ represents initial deviations of the surface in its unstressed state from an ideal shape in which the external loads give rise to only a membrane state of stress, $w = w(x, y)$ is the additional displacement in consequence of elastic and creep deformations, ρ is a scalar factor defining the intensity of the external loads, ε_{ij} is the ijth component of the strain tensor, σ_{ij} is the ijth component of the stress tensor, s_{ij} is the ijth component of the tensor of stress deviation defined as

$$s_{ij} = \sigma_{ij} - (1/3)\sigma_{kk}\delta_{ij} \tag{5}$$

repeated letter subscripts indicate summation over the values 1–3 and thus the average principal stress is

$$\sigma_{av} = (1/3)\sigma_{kk} = (1/3)(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{6}$$

the Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 \text{ when } i = j \\ 0 \text{ when } i \neq j \end{cases} \tag{7}$$

the second invariant of the stress deviation tensor is

$$J_2 = (1/2)s_{ij}s_{ji} \tag{8}$$

G is the shear modulus characterizing linear elastic deformations, v Poisson's ratio, t the time lapsed from the instant when the loads were applied, C, m and q are constants characterizing the creep deformations, Φ, ψ and Ω represent functional relations and the dot above symbols indicates a differentiation with respect to time. The numerical value of m is usually between 1 and 7, and that of q is usually 1/3.

THE CASE OF PURELY ELASTIC DEFORMATIONS

The governing equations simplify considerably when the deformations are purely elastic. It will now be assumed that in the case when the constitutive equation (4) is

$$\varepsilon_{ij} = (1/2G)\left[s_{ij}+\frac{1-2v}{3(1+v)}\sigma_{kk}\delta_{ij}\right] \tag{9}$$

the governing equations can be reduced to the appropriate boundary conditions and to the single differential equation

$$Fw+\rho Hw_0+\rho Hw = 0 \tag{10}$$

where F and H are linear differential operators which may, or may not, involve multipliers dependent on x and y. Examples of such equations are the plate equation (see, for instance, equation (250) on p. 394 of Timoshenko's classical book [6]) and Donnell's equation of the circular cylindrical shell (which is easy to obtain from equation (6) of one of Donnell's classical papers [7]).

It is further assumed that when the structural element is perfect,

$$w_0(x, y) = 0 \tag{11}$$

there exist eigenvalues $\rho = \rho_{cr}$ and a corresponding complete set of eigenfunctions

$$w = f(x, y) \tag{12}$$

for which both the boundary conditions and (10) are identically satisfied when $w_0 = 0$:

$$Fw+\rho Hw = 0. \tag{13}$$

This implies that

$$\begin{aligned} Ff(x, y) &= pg(x, y) \\ Hf(x, y) &= -rg(x, y) \end{aligned} \tag{14}$$

where p and r are scalars. It follows then that the eigenvalue is

$$\rho_{cr} = p/r. \tag{15}$$

Let it be assumed now that the imperfect shell is characterized by

$$w_0 = af(x, y) \tag{16}$$

where a is the amplitude of the eigenfunction $f(x, y)$. The solution of (10) can then be written in the form

$$w = bf(x, y). \tag{17}$$

Indeed, substitution in (10) yields the algebraic equation

$$pb - \rho ra - \rho rb = 0 \tag{18}$$

from which the amplitude b of the deformations caused by the loads follows as

$$b = a\frac{\rho r}{p - \rho r}. \tag{19}$$

Division of numerator and denominator by r and consideration of (15) lead to the expression

$$b = a\frac{\rho}{\rho_{cr} - \rho}. \tag{20}$$

It can be seen that the amplitude of the additional displacements of an initially imperfect structure increases beyond all bounds as the intensity of loading approaches the critical value. One should also note that the amplitude $a+b$ of the total displacements is

$$a+b = a\frac{\rho_{cr}}{\rho_{cr} - \rho}. \tag{21}$$

THE CASE OF PURELY STEADY-STATE CREEP DEFORMATIONS

In this case the constitutive equation (4) is reduced to

$$\dot{\varepsilon}_{ij} = CJ_2^m s_{ij} \tag{22}$$

which is known as Odqvist's law. After a load system of prescribed intensity ρ has been applied, any initial deviation $w_0(x, y)$ is augmented with time. As the creep deformations are considered to be permanent, the natural, unstressed state of the shell at t, which can be designated as $w_0(x, y, t)$, is, in general, different in shape and magnitude from $w_0(x, y, 0)$ at the moment of load application. The first problem of the creep buckling analysis can thus be stated as the determination, from equations (1)–(3) and (22), of the creep deformation rate $\dot{w}(x, y, t)$ at t when the unstressed state of the shell is characterized by $w_0(x, y, t)$.

The problem is a difficult one because of the nonlinearity of (22) when $m > 0$. No rigorous, closed form solution of any problem of this kind is known to the authors. The usual approach to an approximate solution consists of assuming $\dot{w}(x, y, t)$ to be proportional to the eigenfunction $f(x, y)$ corresponding to the lowest eigenvalue of the eigenvalue problem obtained by replacing (22) with (9), and satisfying the governing equations in some approximate manner, for instance, by the collocation method. An improvement on this method of solution consists of expanding the solution into a series of the eigenfunctions with unknown coefficients and requiring that the multipliers of each eigenfunction vanish.

It is of great importance to engineers to know that the ratio of the coefficient of the second eigenfunction to that of the first eigenfunction has been found very small during the major part of the creep process in all those cases in which it was investigated. Such studies were carried out for columns in 1954 [8] and for flat rectangular plates in 1969 [9]. In addition, the results of the theory have been found to agree with the results of experiments in an engineering approximation. It can be concluded, therefore, that it is permissible to base the analysis on the assumption that both the elastic and the creep deformations have the shape of the eigenfunction corresponding to the lowest eigenvalue.

As at any given time t the deflection rate of the column depends only on the state of deflections, one can write

$$\dot{w}(x, y) = \dot{c}f(x, y) = h(c)f(x, y) \tag{23}$$

where $f(x, y)$ is the eigenfunction of the linearly elastic problem corresponding to the lowest eigenvalue and $h(c)$ is a function of the amplitude alone when ρ is prescribed.

The second problem of the creep buckling analysis is the integration of (23). Formally one may write

$$\int \mathrm{d}t = \int \frac{\mathrm{d}c}{h(c)}. \tag{24}$$

In all the problems solved up to now for $m > 0$ a finite time t_{cr} was found at which the deformations tended to infinity. Hence the critical time t_{cr} can be defined as

$$t_{cr.st} = \int_{c_0}^{\infty} \frac{\mathrm{d}c}{h(c)} \tag{25}$$

where c_0 is the magnitude of c at the time when the loads are applied and $t_{cr.st}$ is the critical time in the presence of steady creep.

SIMULTANEOUS NONLINEAR TIME-HARDENING CREEP AND LINEARLY ELASTIC DEFORMATIONS

Equation (25) gives the critical time of the structure when the only mechanism of deformation is steady creep. When the creep deformations are of the time-hardening type, (22) is replaced by

$$\dot{\varepsilon}_{ij} = Ct^q J_2^m s_{ij}. \tag{26}$$

Here t^q can be considered as a constant multiplier of C in all the differentiations and integrations. Thus inclusion of this power of time in the analysis simply leads to

$$\int t^q \,\mathrm{d}t = \int \frac{\mathrm{d}c}{h(c)} \tag{27}$$

in the place of (24). It follows then that the critical time for time-hardening creep can be calculated from

$$\frac{1}{q+1} t_{cr.t.h}^{q+1} = t_{cr.st} \tag{28}$$

where $t_{cr.t.h}$ is the critical time in the presence of time-hardening creep.

When, in addition, the structure is capable of linear elastic deformations, every small increment $\dot{c}\,\mathrm{d}t$ in the creep deformations is magnified by the factor $\rho_{cr}/(\rho_{cr} - \rho)$ in agreement with (21). Thus $h(c)$ in the last member of (23) is multiplied by this factor which remains a constant during the integration with respect to time provided the intensity of the loading remains constant. It follows then that the critical time t^*_{cr} in the presence of simultaneous time-hardening creep and elastic deformations is

$$t^*_{cr} = \left[(q+1)\frac{p_{cr}-p}{p_{cr}}t_{cr.st}\right]^{1/(q+1)}. \tag{29}$$

GENERALIZATION

The arguments presented are valid if Odqvist's law of (22) is replaced by some other creep law that expresses the strain rate as a function of the stresses, and when the time function in (4) is not a power function. It is only necessary that the creep rate should be given as a function of stress multiplied by a function of time. Of course, equations (26)–(29) must be modified if t^q is replaced by some other function.

EXAMPLE

To illustrate the statements made, the simple example of the idealized column of constant cross section simply supported at its two ends and loaded in uniform axial compression will now be worked out. At first it will be assumed that the material of the column deforms only linearly elastically. The notation is shown in Figs. 1 and 2; the analysis is similar to that presented in [3].

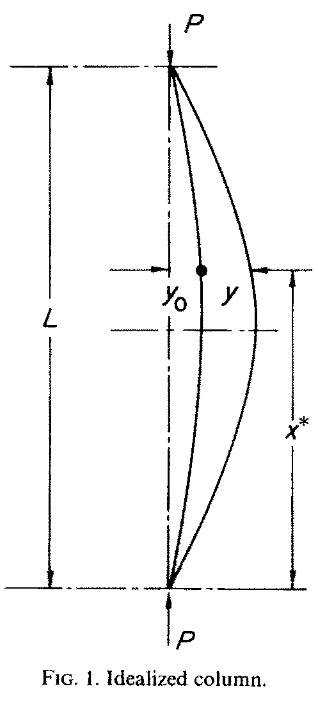

FIG. 1. Idealized column.

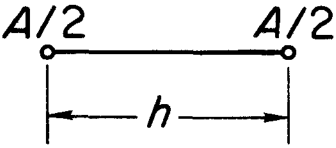

FIG. 2. Cross section of idealized column.

The conditions of equilibrium are:

$$(A/2)(\sigma_i+\sigma_e) = P \tag{30}$$

$$(Ah/4)(\sigma_i-\sigma_e) = P(y_0+y) \tag{31}$$

where σ_i is the stress in the internal (concave) flange of the idealized column, σ_e is the stress in the external (convex) flange of the column and both are counted positive if they are compressive; P is the axial load, also positive if compressive; $A/2$ is the cross-sectional area of one flange; y_0 and y are the small initial and additional lateral displacements.

If the notation

$$w = \frac{y}{h/2} \qquad w_0 = \frac{y_0}{h/2} \tag{32}$$

is introduced, where h is the distance between the flanges of the idealized column, the equilibrium equations can be written in the form

$$\sigma_i = (P/A)(1+w_0+w) \qquad \sigma_e = (P/A)(1-w_0-w). \tag{33}$$

These two equations take the place of (1) in the general case.

The small change $\Delta\kappa$ in the curvature of the column during the transition from the initial shape y_0 to the shape y_0+y is

$$\Delta\kappa = -\partial^2 y/\partial x^{*2} \tag{34}$$

where x^* is the distance from one of the supports. With the notation

$$x = x^*/L \tag{35}$$

equation (34) can be written in the non-dimensional form

$$\Delta\kappa = -(h/2L^2)(\partial^2 w/\partial x^2). \tag{36}$$

But the curvature can also be expressed in terms of the strains ε_i on the internal (concave) and ε_e on the external (convex) side of the column (the strains are considered positive when they are compressive):

$$\Delta\kappa = (1/h)(\varepsilon_i-\varepsilon_e). \tag{37}$$

Hence the strain–displacement relation (2) becomes

$$(\partial^2 w/\partial x^2)+(2L^2/h^2)(\varepsilon_i-\varepsilon_e) = 0. \tag{38}$$

This corresponds to (2) in the general derivation.

The boundary conditions can be given in the form

$$w = (\partial^2 w/\partial x^2) = 0 \quad \text{when } x = 0, 1 \tag{39}$$

and the constitutive equation (4) in the case of exclusively elastic uniaxial deformations reduces to

$$\varepsilon = \sigma/E. \tag{40}$$

Substitution from (33) and (40) in (38) yields the differential equation

$$(\partial^2 w/\partial x^2) + PK^2 w_0 + PK^2 w = 0 \tag{41}$$

where

$$K^2 = 4L^2/EAh^2. \tag{42}$$

This is (10) if $F = \partial/\partial x^2$ and $\rho H = PK^2$.

When the column is perfectly straight,

$$w_0 = 0 \tag{43}$$

and (41) reduces to

$$(\partial^2 w/\partial x^2) + PK^2 w = 0 \tag{44}$$

which corresponds to (13) in the earlier derivations. This equation and boundary conditions (39) are satisfied by

$$w = \sin \pi x = f(x, y). \tag{45}$$

From (13) and (14) one obtains

$$\begin{aligned} Fw &= -\pi^2 \sin \pi x = pg(x, y) \qquad \rho = P \\ Hw &= K^2 \sin \pi x = -rg(x, y) \end{aligned} \tag{46}$$

and (15) yields the critical value of the axial load

$$P = P_{cr} = p/r = \pi^2/K^2 = \frac{\pi^2 EA(h/2)^2}{L^2} = \frac{\pi^2 EI}{L^2} = P_E \tag{47}$$

which is the lowest eigenvalue of the homogeneous differential equation (44) in the presence of boundary conditions (39). It is known as the Euler load. The quantity $I = A(h/2)^2$ is the moment of inertia of the section of the idealized column.

It follows then from (17) and (21) that in the presence of initial deviations w in the shape of the first eigenfunction (45) the sum of the initial and the additional deformations becomes

$$w = a\frac{P_E}{P_E - P}\sin \pi x. \tag{48}$$

Next the case of purely steady-state creep deformations will be investigated. Under these conditions (4) simplifies to

$$\dot{e} = \dot{\varepsilon} = k\sigma^n \tag{49}$$

with

$$n = 2m+1 \qquad k = 2C/3^{m+1}. \tag{50}$$

Use will be made of the elastic analogy of [10] and the instantaneous deflected shape at t as caused by elastic and creep deformations will be designated by the symbol w_{tot}. It will be represented by

$$w_{tot} = \gamma \sin \pi x \tag{51}$$

which is the eigenfunction corresponding to the lowest eigenvalue of the elastic problem multiplied by the coefficient γ. When the axial load P is acting on this analogous deflected column, equilibrium requires that

$$\sigma_i = (P/A)(1+w_{tot}) \qquad \sigma_e = (P/A)(1-w_{tot}). \tag{52}$$

The additional deformation w_{ad} of this non-linearly elastic column will be assumed to be represented by the first term of the infinite series

$$w_{ad} = c \sin \pi x + \ldots . \tag{53}$$

The deformation law of the analogous column is

$$\varepsilon = k\sigma^n. \tag{54}$$

On the basis of (38) the strain–displacement relation of the analogous column is

$$(\partial^2 w_{ad}/\partial x^2)+(2L^2/h^2)(\varepsilon_i-\varepsilon_e) = 0. \tag{55}$$

If n is taken as 3, it follows from (52) and (54) that

$$\varepsilon_i-\varepsilon_e = 2k(P/A)^3(3w_{tot}+w_{tot}^3). \tag{56}$$

But

$$(\sin \pi x)^3 = (3/4)\sin \pi x-(1/4)\sin 3\pi x. \tag{57}$$

Since the terms $\sin 3\pi x$, $\sin 5\pi x, \ldots$ were omitted from equation (53), the present analysis deals only with multipliers of $\sin \pi x$. Thus (56) becomes†

$$\varepsilon_i-\varepsilon_e = 2k(P/A)^3[3\gamma+(3/4)\gamma^3]\sin \pi x. \tag{58}$$

Substitution from (53) and (58) in (55) and omission of the common trigonometric multiplier yield

$$-\pi^2 c+6k(2L^2/h^2)(P/A)^3[\gamma+(1/4)\gamma^3] = 0. \tag{59}$$

This can be solved for the amplitude c of the additional displacements of the analogous elastic column:

$$c = (6k/\pi^2)(2L^2/h^2)(P/A)^3[\gamma+(1/4)\gamma^3]. \tag{60}$$

Symbols will now be introduced for the following physical quantities:

$$\left.\begin{aligned}
\sigma_E &= P_E/A = \frac{\pi^2 E}{4(L/h)^2} && \text{is the Euler stress}\\
\varepsilon_E &= \sigma_E/E = \frac{\pi^2}{4(L/h)^2} && \text{is the Euler strain}\\
\dot{\varepsilon}_{nom} &= k\sigma^3 = k(P/A)^3 && \text{is the nominal (initial) strain rate}\\
t_E &= \varepsilon_E/\dot{\varepsilon}_{nom} && \text{is the Euler time.}
\end{aligned}\right\} \tag{61}$$

† In a similar manner, the higher harmonics should be dropped when the displacements are expressed by double Fourier series, or by some other orthonormal double series, in the case of two-dimensional problems.

With these symbols (60) can be written in the form

$$c = (3/4)(1/t_E)(4\gamma+\gamma^3). \tag{62}$$

This is the amplitude of the additional displacements of the nonlinearly elastic analogous column. But in view of the elastic analogy this is also the rate of change of the amplitude of the creep deformations of the column that deforms nonlinearly in consequence of steady creep. Thus for the column that deforms only in steady creep from the state characterized by w_{tot} according to (51) the rate of change of the creep deformations becomes on the basis of (51) and (53)

$$\dot{w}_{tot} = \dot{\gamma} \sin \pi x = (3/4)(1/t_E)(4\gamma+\gamma^3) \sin \pi x. \tag{63}$$

Since $\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}t$, if the only mechanism of deformation were steady creep, the time required to reach a deformed state characterized by the amplitude γ_1 would be given by the integral

$$t_1 = (4/3)t_E \int_{\gamma_0}^{\gamma_1} \frac{\mathrm{d}\gamma}{4\gamma+\gamma^3} = (1/6)t_E \lg\left(\frac{\gamma_1^2(4+\gamma_0^2)}{\gamma_0^2(4+\gamma_1^2)}\right). \tag{64}$$

The critical time of creep buckling is defined as the time at which the amplitude c_1 approaches infinity. Hence

$$t_{cr} = (1/6)t_E \lg\left(\frac{4+\gamma_0^2}{\gamma_0^2}\right). \tag{65}$$

This is the formula obtained for this case in [3].

When the creep law is the time-hardening law

$$\dot{\varepsilon} = k\sigma^n t^q \tag{66}$$

the effect of time hardening can be followed up by replacing k with kt^q in every step of the foregoing derivations. It is easy to see that in such a case (63) is replaced by

$$\dot{\gamma} = \partial\gamma/\partial t = (3/4)(t^q/t_E)(4\gamma+\gamma^3). \tag{67}$$

The integral of this equation is

$$\int_0^t t^q \,\mathrm{d}t = \frac{t^{q+1}}{q+1} = (4/3)t_E \int_{\gamma_0}^{\gamma_1} \frac{\mathrm{d}\gamma}{4\gamma+\gamma^3} = (1/6)t_E \lg \frac{\gamma_1^2(4+\gamma_0^2)}{\gamma_0^2(4+\gamma_1^2)}. \tag{68}$$

The effect of linear elastic deformations can also be taken into account without difficulty. Whenever creep causes an increase $\mathrm{d}\gamma$ in the amplitude of the deformations, this increase is magnified by the factor $P_E/(P_E-P)$ because of the elasticity of the material. Since this factor is constant with time, (67) has to be modified to read

$$\mathrm{d}\gamma/\mathrm{d}t = [P_E/(P_E-P)](t^q/t_E)(4\gamma-\gamma^3) \tag{69}$$

and the time to reach the prescribed displacement amplitude γ_1 becomes

$$[1/(q+1)]t^{q+1} = (1/6)[(P_E-P)/P_E]t_E \lg \frac{\gamma_1^2(4+\gamma_0^2)}{\gamma_0^2(4+\gamma_1^2)}. \tag{70}$$

The critical time t_{cr}^* of the column whose material deforms simultaneously because of time-hardening creep and linear elasticity can thus be calculated from the equation

$$[1/(q+1)]t_{cr}^{*q+1} = (1/6)[(P_E-P)/P_E]t_E \lg \frac{4+\gamma_0^2}{\gamma_0^2} \quad (71)$$

or from

$$t_{cr}^* = \left\{(q+1)(t_E/6)[(P_E-P)/P_E] \lg \frac{4+\gamma_0^2}{\gamma_0^2}\right\}^{1/(q+1)}. \quad (72)$$

This expression corresponds to (29). It also agrees with the result of the analysis of [3] if q is set equal to zero; this is necessary since the calculations of [3] were carried out for a column that deforms elastically and because of steady creep.

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Абстракт—Указано, что выпучивание при ползучести гибких или тонкостенных конструкций, материал которых деформируется в упругой области также вследствие ползучести связанной с упрочнением во времени, можно обсуждать в первых как деформации, вызванные полностью вследствие установившейся стадии ползучести. Результаты предлагаемшго анализа можно легко модифицчровать с целью учета ползучести упрочнения во времени. Такая модификация не требует никакого приближения для случая, когда скорость деформации ползучести упрочнения во времени выражена произведением функции напряжения умноженной на функцию времени.

Эффект совместных линейно упругих деформаций можно учитывать, умножая критическое время, полученное для установившейся ползучести на численный фактор. Эта корректура не требует никаких далее достигающих приближений как обыкновенный анализ выпучивая для случая уснановившейся ползучести.